Journal of Physics: Conference Series **3280** (2026) 012016 doi:10.1088/1742-6596/3280/1/012016

Computational Modeling, a case study: the Wilberforce pendulum

Giacomo TORZO [1,*], **Michele D'ANNA**[2] and **Massimiliano ZANICHELLI** [3]

[1] AIF-Labtrek and Dept. of Physics, Padova University, Padova, Italy
[2] Liceo cantonale, Locarno, Switzerland (retired)
[3] IISS Emilio Gadda, via Nazionale 6, Fornovo, Italy

*E-mail: torzog@gmail.com

Abstract. We use a Wilberforce pendulum with a special MEMS sensor to present an interesting example of Computational Modeling employing either the free App Phyphox for smartphones/tablets plus the web-based modeling tool InsightMaker, or the integrated Coach7 system for both data-acquisition and modeling, to show how a detailed analysis of the motion and of the energetical aspects may be easily performed.

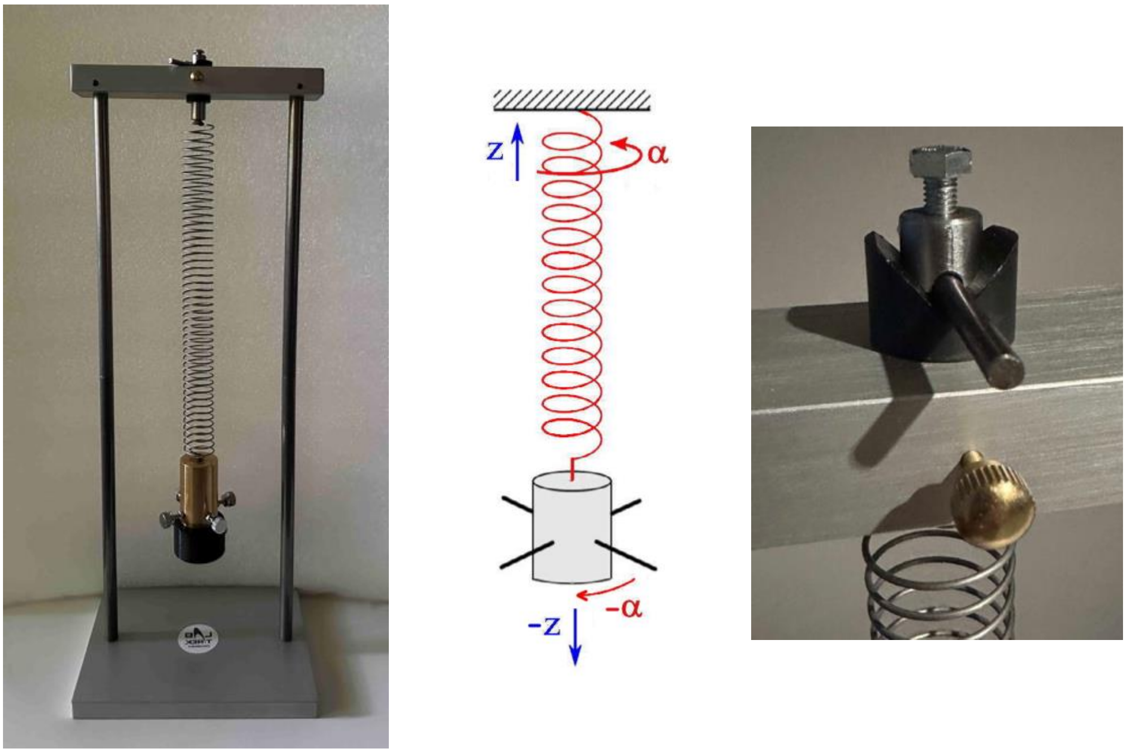
Introduction

The Wilberforce pendulum [1] is a didactical device often used in classroom demonstrations to illustrate coupled oscillations (rotational and longitudinal), producing fascinating beat phenomena when the system consisting in an object attached to a simple helical spring is properly tuned.

It is a system that has been extensively studied in the literature ([2–6] and references therein), both from the experimental and the theoretical points of view. The formal solution, however, requires dealing with a set of coupled second-order differential equations, a task that goes beyond the mathematical skills of upper secondary school students, to whom this work is addressed. The aim of the present contribution is precisely to show how modern technology and contemporary tools for dynamic modeling allow students at this educational level to obtain a satisfactory characterization of the observed motion, while bypassing these formal difficulties.

1. Setup and experimental results

Our pendulum [6] uses an helicoidal spring and a brass cylindrical bob with 4 screws for fine-tuning of the inertial moment, and a MEMS absolute orientation sensor (including accelerometer and gyroscope) driven by an ESP32 microcontroller with Bluetooth and Li-ion battery charger (figure 1). The top end of the spring is fixed to a brass cylinder that may both rotate along the vertical axis and shift in vertical direction. By a proper choice of initial displacement z_0 and rotation angle α_0 we may excite each one of the three different pendulum motions we are interested in.



a) b) c)

Figure 1. The Wilberforce pendulum: a) complete experimental set up; b) used reference frame; c) detail of suspension tool (allowing excitation of normal modes).

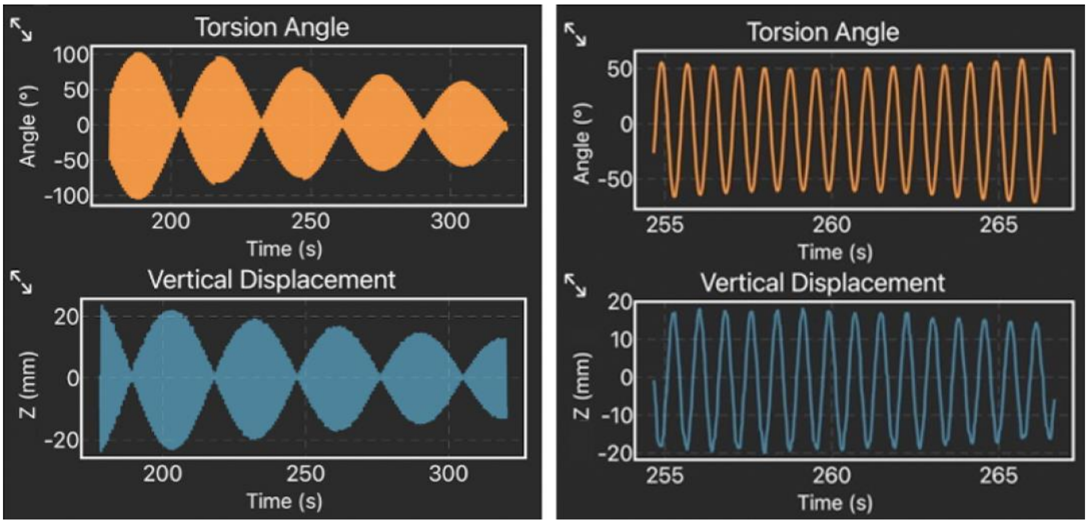
The parameters m, k, I and δ may be measured directly or calculated from measured motion. To determine k we may measure the spring elongation under a known force (e.g. the cylinder weight mg). The mass of the pendulum bob (m), the spring constant (k), the inertial moment (I) and the torsional spring constant (δ) may be measured directly or calculated from measured motion. To determine k we may measure the spring elongation under a known force (e.g. the cylinder weight mg).
A measurement of the moment of inertia I for our *non-cylindrical* bob (consisting of the brass cylinder, its protruding screws, and the box containing the electronics attached at the bottom) may be obtained through two measurements of the rotational period: first with the bob only, and then by adding an *annular cylinder* (of known mass M, inner radius R_1, outer radius R_2), whose moment of inertia is $I_a=M(R_2^2+R_1^2)/2$. The two periods (without and with the annular cylinder) obey to the relations: $(T/2\pi)^2=I_x/\delta$, $(T_1/2\pi)^2 = (I_x+I_a)/\delta$, giving: $I_x= I_aT^2/(T_1^2-T^2)$.

We can finally determine the *torsion constant* δ from the measured period $T=2\pi/\omega$: $\delta = I(2\pi/T)^2$.

As is well known, from an operational standpoint, the release of the pendulum after a slight downward displacement (with no rotation of the bob) results in the characteristic motion of the Wilberforce pendulum, namely the periodic transfer between vertical oscillatory motion and rotational motion about the cylinder axis. Our apparatus is equipped with a device for the excitation of the normal modes (Figure 1c): by rapidly rotating the lever either clockwise or counterclockwise, in addition to the vertical displacement, a torsional deformation can be induced, thereby placing the system in a state that corresponds to the specific initial conditions of the normal modes.

Journal of Physics: Conference Series **3280** (2026) 012016 doi:10.1088/1742-6596/3280/1/012016

Figure 2 shows data taken using Phyphox and smartphone for a motion excited in mixed mode (with beating) and for a motion excited in the antisymmetric normal mode.



a) b)

Figure 2. Some examples of experimental results recorded with Phyphox/smartphone . Initial conditions: a) $z_0 \neq 0$ and $\propto_0 = 0$ give beatings; b) $\propto_0 = -z_0/\Gamma$ anti-symmetric normal mode with frequency ω_0.

2. A simplified physical model

The behaviour of the system is governed by the two dynamical laws of motion: one describing translational motion and the other describing rotational motion. The former can be explicitly formulated when all the acting forces are known, while the latter can be formulated once all the acting torques are known.

Restricting ourselves in this section to the ideal case (absence of damping), the translational dynamics is determined by three contributions: the gravitational force, the elastic force of the spring, and the interaction force responsible for the coupling between translational and rotational motion. This interaction force is chosen according to the model proposed by Sommerfeld [2]. For the level of accuracy required here, the buoyant force due to the presence of air can be neglected.

As far as rotational motion is concerned, only two torques are present: the elastic restoring torque and the torque generated by the elongation of the spring, which represents the rotational counterpart of the interaction force.

With the choice indicated above (z=0 at equilibrium), the equations of motion can be written as follows:

$$
\begin{aligned}
m\frac{d^2z}{dt^2} &= -kz - \frac{\epsilon}{2}\alpha \\
I\frac{d^2\alpha}{dt^2} &= -\delta\alpha - \frac{\epsilon}{2}z
\end{aligned}
\qquad (1)
$$

The fine-tuning condition requires the equality of the frequencies ω_z and ω_α that separately characterize the translational and rotational motions (in the hypothetical absence of coupling), namely:

$$\omega_z = (k/m)^{1/2} = \omega_\alpha = (\delta/I)^{1/2} = \omega_0. \tag{2}$$

As explicitly shown in [3, 6], by exploiting the fact that the beating frequency is much larger than the mechanical oscillation frequency (see Fig. 2a), the following approximate expressions for the eigenfrequencies of the system are obtained:

$$\begin{aligned} \omega_1 &= \omega_0 + \varepsilon / (16 k I)^{1/2} \\ \omega_2 &= \omega_0 - \varepsilon / (16 k I)^{1/2}\,. \end{aligned} \tag{3}$$

These expressions show *a posteriori* that the approximations employed are valid in the so-called weak-coupling regime. They also allow one to infer that the beating frequency ω_B is given by:

$$\omega_B = \omega_1 - \omega_2 = \varepsilon / (4 k I)^{1/2} \quad , \tag{4}$$

an interesting relation, since the experimental determination of the eigenfrequencies (for instance by analysing the oscillation spectrum) allows one to determine the value of the coupling constant ε.

We note that it is also possible to show that the normal modes (i.e., without beatings) can be excited by appropriately choosing the *initial conditions* $z_0 = \pm\Gamma\alpha_0$, where $\Gamma = (I/m)^{1/2}$ is the *radius of gyration* of the pendulum.

Concerning the energetic aspects, the relation giving the total energy of the Wilberforce pendulum can be written as the sum of a rotational term E_R a translational term E_T (both including kinetic and elastic contributions), and a coupling term E_ε:

$$E_{Wilber} = E_T + E_R + E_\varepsilon = (mv^2 + kz^2 + I\omega^2 + \delta\alpha^2 + \varepsilon\alpha z)/2 \tag{5}$$

When damping needs to be considered, the situation becomes considerably more complicated from an analytical point of view. As we will see in the next section, however, its presence can be easily handled using dynamic modeling tools.

3. Dynamical Modeling

The dynamic modeling approach proposed here not only avoids requiring high school students to use mathematical skills beyond their reach but also aims to provide them with a framework in which they can build a direct physical understanding of the various processes at play. The approach is based on three fundamental pillars: a) the identification of the linear momentum p_z of the pendulum bob and its angular momentum L (with respect to its center of mass) as *extensive quantities* characterizing the processes under consideration; b) the knowledge of *capacitive laws* that describe how variations of the extensive quantities within the system are linked to variations of the conjugate intensive quantities—velocity and angular velocity in our case; c) the *kinematic relations* that allow velocities to be interpreted as rates of change of the corresponding positions.

Since, in the case under consideration, the extensive quantities involved are conserved quantities, for each of them it is possible to directly relate the instantaneous rate of change to the sum of all exchanges occurring at that instant between the system and the surrounding environment. The relations expressing this equality can therefore be interpreted as *balance equations* for the corresponding quantities, thus providing, in our case, the possibility of easily incorporating any form of friction.

Journal of Physics: Conference Series **3280** (2026) 012016 doi:10.1088/1742-6596/3280/1/012016

From a formal point of view, this means that each of equations (1) is replaced by a pair of first-order differential equations, which are easier to handle (generally through numerical integration) and more straightforward to interpret from a physical standpoint. In our case, we obtain:

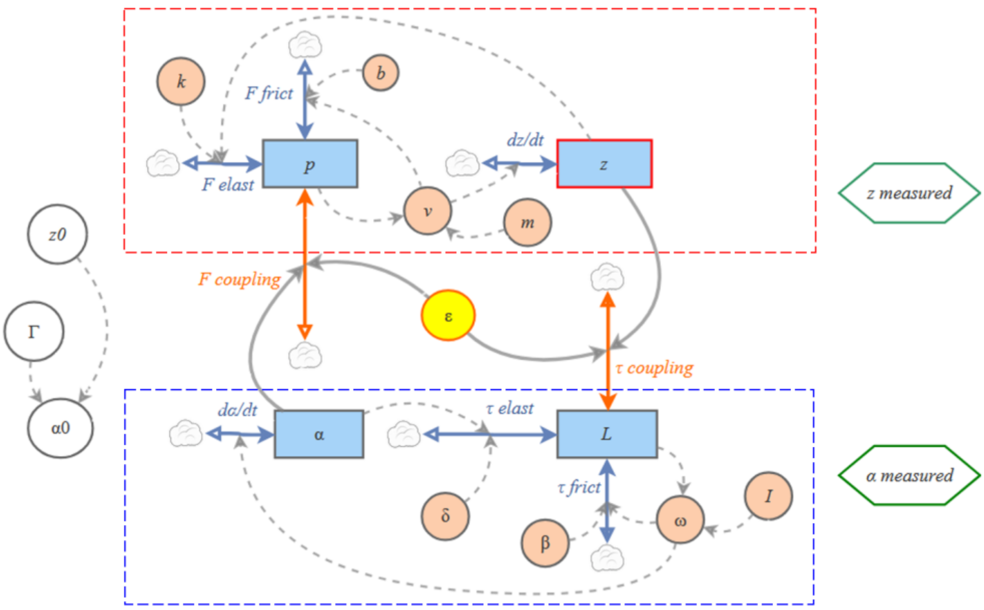
$$\begin{cases} v_z = \frac{dz}{dt} \\ \frac{dp_z}{dt} = -kz - \frac{\epsilon}{2}\alpha - bv_z \end{cases} \qquad \begin{cases} \omega = \frac{d\alpha}{dt} \\ \frac{dL}{dt} = -\delta\alpha - \frac{\epsilon}{2}z - \beta\omega \end{cases} \tag{6}$$

where, in order to account for dissipative effects in the two balance equations, a viscous damping force F_{frict} and a viscous damping torque τ_{frict} have been introduced, respectively.

Equations (6) completely determine the time evolution of the system. Several approaches can be used to describe the energetic aspects: for example, the instantaneous values of the kinetic and elastic energies can be expressed using the values of the velocities and positions at that instant (together with the inertia and elasticity parameters). To obtain the value of the dissipated energy E_{diss} at time t (both for translational and rotational motion), a different procedure is required. First, the instantaneous dissipation rate as a function of time—namely, the *dissipated power* as a function of time—must be expressed; this quantity is then integrated from the beginning of the process up to the time t under consideration:

$$\begin{array}{ll} P_{dissT}(t) = v_z(t)\left|F_{frict}(t)\right| = b\, v_z^2(t) & P_{dissR}(t) = \omega(t)\left|\tau_{frict}(t)\right| = \beta\omega^2(t) \\ E_{dissT}(t) = \int_0^t P_{dissT}(t')\,dt' = \int_0^t b\, v_z^2(t')dt' & E_{dissR}(t) = \int_0^t P_{dissR}(t')\,dt' = \int_0^t \beta\,\omega^2(t')dt' \end{array} \tag{7}$$

From a technical perspective, both in *InsightMaker* [8] and *Coach7* [9] we create the model using: a) "*reservoirs*" which represents physical quantities (extensive quantities to be "stored" in the system, as momentum *p=mv*, angular momentum *L=I*ω, energy); b) "*flow rates*", which implement, at each instant, the value of the incoming and/or outgoing exchanges of the reservoir to which they are connected; c) "*constants* or *variables*", which represent general physical quantities; d) "*links*", to connect the various elements.



Journal of Physics: Conference Series **3280** (2026) 012016 doi:10.1088/1742-6596/3280/1/012016

Figure 3: InsightMaker dynamical model for the Wilberforce pendulum. The graphical structure highlights the coupling between the two oscillators (translational and rotational). The model is available at https://insightmaker.com/insight/7nEQFGStJR61YFKeyXlmHN/GIREP-2025-Wilberforce-beats.

Figure 3 shows the model developed with InsightMaker, including all the elements that determine the dynamical evolution of the vertical position and the rotation angle. The red box contains the elements describing the translational oscillation, while the light-blue box includes those related to the rotational oscillation. These two "oscillators" are coupled through the interaction described above (solid grey lines). A detailed description of this model can be found in [6].

Figure 4 shows the comparison between the measured data for the vertical position z and the values calculated by the model for two different situations (in InsightMaker, it is necessary to import the data recorded by Phyphox as a CSV file). Good agreement is achieved by optimizing the values of the parameters b and β, which control the damping.

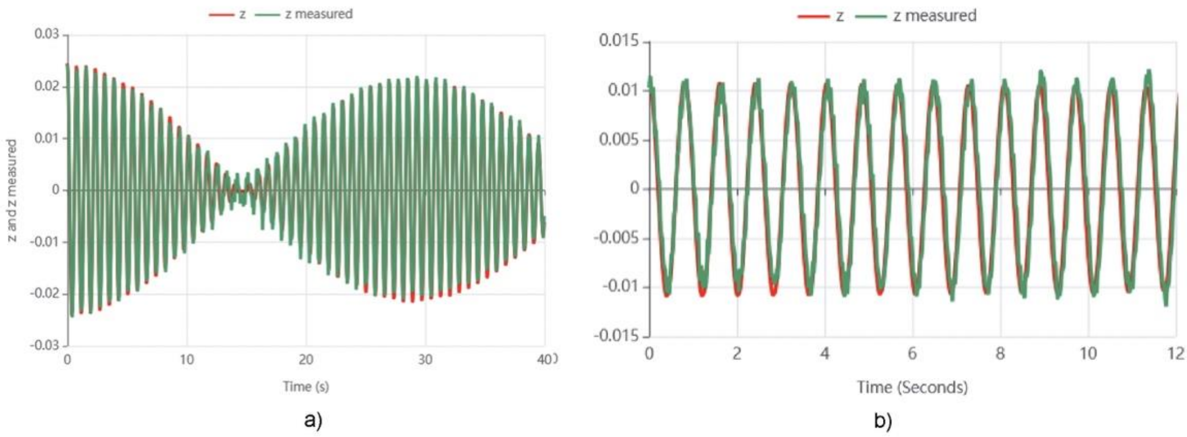


Figure 4: Comparison between model results (red line) and experimental data (green line): a) for beating condition: the agreement is such that the two curves are almost indistinguishable.; b) for the antisymmetric normal mode: the small modulation observed in the collected data reflects a slight imperfection in the preparation of the initial excitation conditions.

Journal of Physics: Conference Series **3280** (2026) 012016 doi:10.1088/1742-6596/3280/1/012016

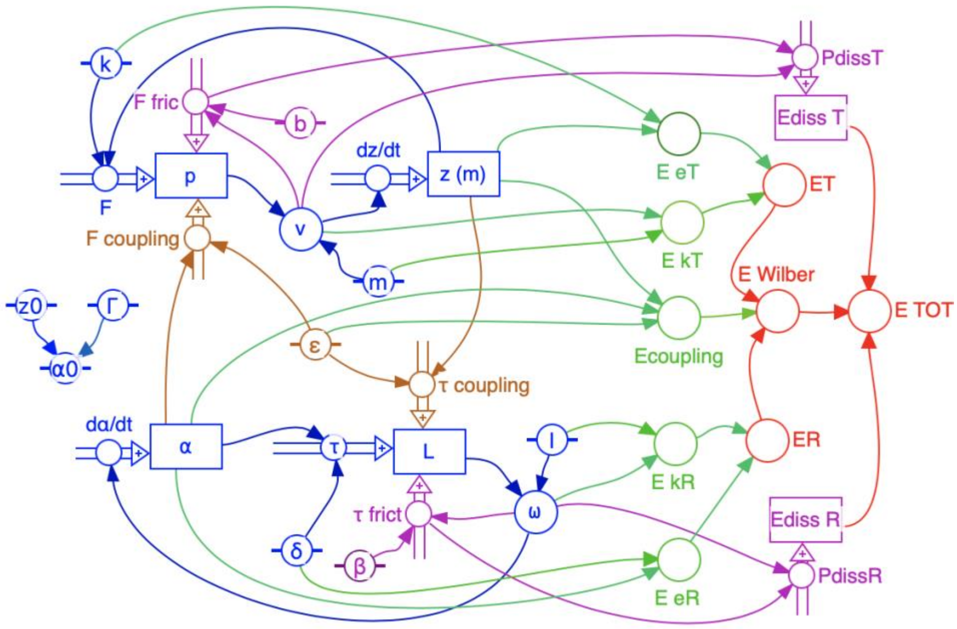


Figure 5: Wilberforce pendulum: the complete model realized within Coach7.

Figure 5 shows the same model extended to the energy aspects created with Coach7: in this working environment data are first recorded with a *Measurement Activity* and then imported into a *Modeling Activity* for comparison.

In addition to the results already discussed above, the model shows to the student that the energy is cyclically exchanged between translation and rotation oscillations, that in both oscillations the energy is converted from *elastic-potential* and *kinetic* terms, that *interaction energy* oscillates with frequency 2ω, with a beating frequency $2\omega_B$, and reaches zero amplitude every time the motion is purely rotational or purely translational. All these aspects of the system-dynamics cannot be easily studied without modeling: figure 6 illustrates some details.

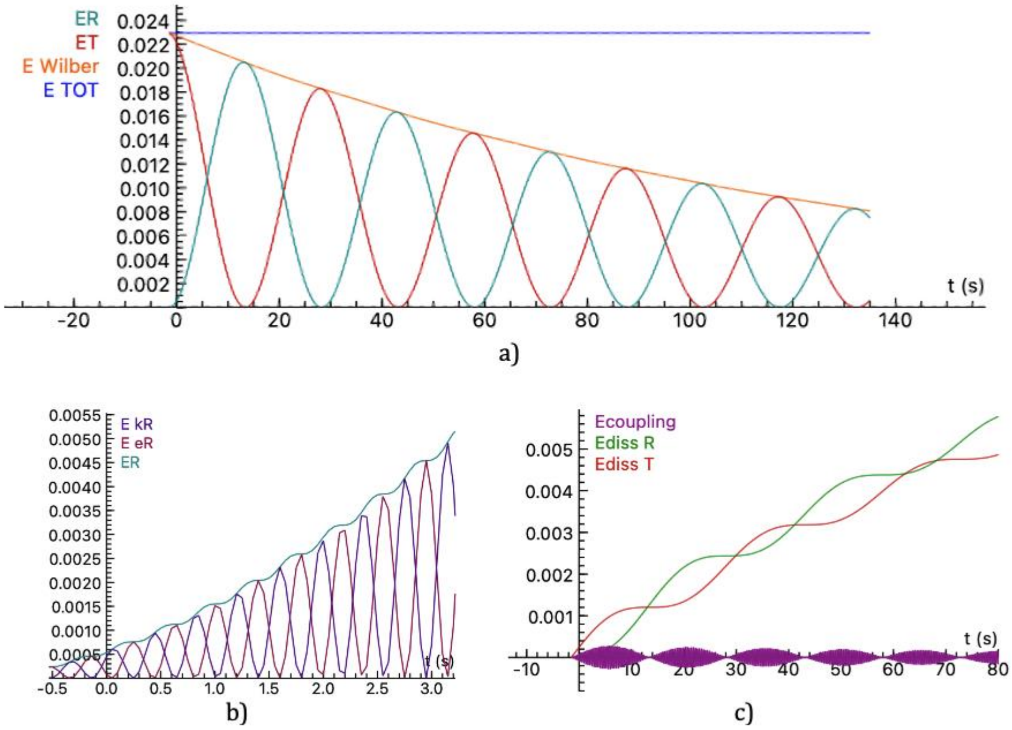


Figure 6: Model results for selected energy aspects: (a) overall view showing the energy associated with translational motion (ET) and rotational motion (ER), the total energy of the pendulum (Ewilber), and the total energy (E TOT); (b) detailed view of the kinetic, elastic, and total energy for the rotational oscillatory motion; (c) detailed view of the energy dissipated in translational motion (red curve) and in rotational motion (green curve), as well as the interaction energy (purple curve).

5. Conclusions

In this contribution, we describe: (a) a novel apparatus that allows the independent measurement of both vertical and rotational displacements, in combination with the Phyphox application or the Coach7 system, and (b) a dynamical modeling approach that makes it possible to conveniently characterize this pendulum and to quantitatively compare the predictions of the model with the collected data.

In this way, the Wilberforce pendulum can be studied both in the classroom and, preferably, as a laboratory activity already at the upper secondary school level. Moreover, the use of dynamical modeling tools such as InsightMaker and Coach not only makes it possible to bypass the formal difficulties arising from the lack of the mathematical tools required to solve a system of coupled second-order differential equations but also offers the opportunity to deepen the analysis of the system from an energetic point of view. In particular, it is possible to go beyond the ideal model by analysing the real situation in detail, explicitly considering the dissipative phenomena responsible for the slow damping observed.

We investigated the time evolution of the vertical and torsional oscillatory motions, the energies associated with these oscillations, as well as the coupling energy and the dissipated energy, also verifying the consistency of the model with the principle of energy conservation.

More generally, this example allows students to be introduced to important concepts such as normal modes, eigenfrequencies, superposition, and beats.

Journal of Physics: Conference Series **3280** (2026) 012016 doi:10.1088/1742-6596/3280/1/012016

References

[1] R. L. Wilberforce (1895) Phil. Mag. **38** ,386
[2] A. Sommerfeld (1905) Adolph Wüllner Festschrift, Leipzig, 162-193
[3] R. E. Berg and T. S. Marshall (1991) Am. J. Phys. **59**, 32-38
[4] U. Köpf, (1990) Am. J. Phys. **58**, 833-837
[5] T. Gallot, D. Gau, R. Garcia-Tejera (2023) Am. J. Phys. **91**, 873-878
[6] G. Torzo, M. Zanichelli, S. Pasqualotto, M. D'Anna (2025) Lat. Am. J. Phys. Educ. Vol. 19, No. 4, 1306
[7] https://phyphox.org
[8] http://insightmaker.com
[9] https://cma-science.nl/coach7_desktop_downloads_en